

Electromagnetic force in dispersive and transparent media

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A hydrodynamic-type, macroscopic theory was derived recently to account for dissipation and dispersion of electromagnetic field, in nonstationary condensed systems of nonlinear constitutive relations [Yimin Jiang and Mario Liu, *Phys. Rev. Lett.* **77**, 1043 (1996)]. As it is published, some algebra and the more subtle of the reasonings had to be left out. They are presented in this paper and include (i) how the results algebraically reduce to the classic ones in the appropriate limits, and (ii) more thoughts on the range of validity of the theory, especially concerning dissipation. [S1063-651X(98)06611-2]

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I. INTRODUCTION

Two theories were recently set up to account for the behavior of condensed media subject to strong external fields. They describe dissipation and dispersion, and yield an explicit expression for the electromagnetic force. The first is a hydrodynamic theory valid for low frequencies [2], and the second a generalization for higher frequencies [1].

Let us first understand how the hydrodynamic theory and the usual theory of linear response complement each other. The macroscopic Maxwell equations, given in terms of the four fields \mathbf{E} , \mathbf{D} , \mathbf{H} , and \mathbf{B} , need constitutive relations linking them to be closed. The form of these relations depends on two physical parameters, the frequency and the field strength. Weak fields are necessary for the linear response theory ($\epsilon\mathbf{E}=\mathbf{D}$, $\mu\mathbf{H}=\mathbf{B}$) to hold; while the hydrodynamic constitutive relations [2] presuppose small frequencies (but not the weakness of fields).

There is a second closure problem one level up that many are not as aware of: A theory of electromagnetism in material is only complete if it also accounts for the feedback, the electromagnetic force on the material. In the microscopic electrodynamics, this is the Lorentz force—the Maxwell equations yield the field produced by the sources, while the Lorentz force (in conjunction with the Newton equation) describes how, conversely, the field changes (the position and motion of) the sources.

In the macroscopic theory, we also need an expression for the force, now in terms of the macroscopic fields, \mathbf{E} , \mathbf{D} , \mathbf{H} , and \mathbf{B} . This information is contained in two related quantities: (i) the additional energy due to the presence of the electromagnetic field, and (ii) the flux of the conserved, total momentum density. The hydrodynamic theory provides unambiguous expressions for both, and is therefore closed and complete even at this second level. Circumstances are less fortunate for the linear response theory, as these two expressions are known here only with a string of additional restrictions.

Assuming transparency (i.e., lack of dissipation), quasi-monochromatic external field and stationarity (i.e., identically vanishing velocity field of the condensed system)—all

in addition to the linearity of the constitutive relations—Brillouin obtained the field energy in 1921, while Pitaevskii, forty years later, arrived at the attendant expression for the total stress tensor, see Landau and Lifshitz [3], and the review article by Kentwell and Jones [4].

If we draw a diagram of field strength versus frequency ω , with the field strength pointing to the right, and ω upward, a vertical stripe along the ω axis represents the range of validity for the linear response theory, while the hydrodynamic theory reigns within a horizontal stripe along the field axis. The expressions of Brillouin and Pitaevskii are valid in isolated patches in the vertical stripe, wherever dissipation is negligible [5].

The parameter space beyond the above two perpendicular stripes needs a theory that is a generalization of both the linear response theory and the hydrodynamic theory, as it must simultaneously account for dissipation, dispersion, nonlinear constitutive relations, and finite velocities. Although one might expect principal difficulties in setting up such a theory, since neither of the two parameters, field and frequency, remains small, we are (up to and maybe slightly beyond the optical frequencies $\approx 10^{15}$ Hz) still in the realm of macroscopic physics, as the electromagnetic wavelength remains large compared to the atomic graininess. And when seeking the expression for the electromagnetic force on a volume element of condensed matter, exerted by (say) a strong laser beam, if we confine our curiosity to the temporally averaged force—with a resolution larger than the time needed to establish local equilibrium—a simple, universal and hydrodynamic-type theory is still possible. Such a theory was derived recently [1] by generalizing the hydrodynamic theory of electromagnetism.

The generalization was accomplished by taking the polarization \mathbf{P} and its temporal derivative $\dot{\mathbf{P}}$ as independent variables, and explicitly deriving the equation of motion for \mathbf{P} . This step eliminates the necessity for frequency dependent constitutive relations, which lies at the root of the difficulties one encounters when attempting to extend the linear response theory into the nonlinear region. (In this first step, the magnetization has not been included as an independent vari-

able. So the theory is only valid for statically nonmagnetic media.)

It is important to realize that the questions posed and answered here are very close to experiments, and hence of great relevance. This is a theory the variables of which are directly measurable: the macroscopic electromagnetic fields, the density, the temperature, and the averaged particle velocity. Consisting of equations of motion, the theory predicts the values of these variables for the next instance if their present values are given. A typical result is the density distribution of particles, and the associated temperature field, in the presence of an oscillating electromagnetic field.

In this paper, we show explicitly that the nonlinear expressions for the energy and the stress tensor indeed reduce to the known ones, of Brillouin and Pitaevskii, in the specified limit. Because of the chosen approach of generalization discussed above, this outcome is by no mean obvious and assured. Besides, the associated algebra is fairly involved and needs to be presented. Once accomplished, this provides two bonuses: It strengthens our trust in the new theory and provides a transparent interpretation for the old and classic results which, having relied heavily on algebra, are somewhat lacking in appropriate physical pictures.

This generalized, hydrodynamic-type macroscopic theory for higher frequencies, or for brevity, *hydrodynamic theory of dispersion*, will be presented in Sec. II, to render the present manuscript self-contained. (Nevertheless, since we shall mostly only list the relevant formulas and abstain from repeating all the reasoning and arguments that lead to the new theory, the reader is advised to also read Ref. [1].) In addition, the question about the range of validity of the new theory is discussed here in greater detail than in [1], at the beginning of the next section.

Section III incorporates the specified approximations and deduces four results. They are compared to the energy density by Brillouin, and to three formulas by Pitaevskii: the total stress tensor, the “nonmagnetic” magnetization, and the time dependent permittivity. Section IV ends with a brief summary.

II. THE HYDRODYNAMIC THEORY OF DISPERSION

In this section, we shall first discuss in some detail the range of validity of the theory, then present its complete set of equations, and specify the theory by an expansion of the thermodynamic energy to third order in the field variables.

A. Range of validity

The proper hydrodynamic theory of electromagnetism [2] accounts for the macroscopic dynamics of continuous media in the low frequency limit, of a system that is charged or exposed to external fields. Local thermodynamic equilibrium holds, and the set of hydrodynamic variables is identical to that of the thermodynamic variables. The equations of motion are conservation laws and the Maxwell equations, including irreversible terms accounting for dissipation.

At higher frequencies, microscopic variables deviate more and more from equilibrium, becoming independent, and finally ballistic. Denoting the time τ_{loc} needed to establish local equilibrium, this starts to happen when $\omega\tau_{loc}$ is no longer small. To account for this circumstance, we usually

have to abandon the hydrodynamic theory and embrace the Boltzmann theory which, despite its undeniable usefulness, is both a far more complex and a less general theory—it considers the vast number of microscopic degrees of freedom explicitly, and it is confined to dilute systems. The question therefore is whether an appropriately generalized hydrodynamic theory can be made to account for some of the more interesting aspects at higher frequencies, and save us from the complexities of the Boltzmann theory.

Let us concentrate on one such microscopic, nonthermodynamic variable, the polarization \mathbf{P} . Actually, as far as its spatial extent is concerned, it is a macroscopic rather than a microscopic variable, but it is certainly a dependent one in the hydrodynamic limit, as long as $\omega\tau_p \ll 1$ holds, where τ_p is the longest of \mathbf{P} 's characteristic times.

In a dielectric medium, \mathbf{P} has many characteristic times, given by its resonance frequencies and their widths. If they are well separated, then the equation of motion for \mathbf{P} , close to one resonance and in the simplest case considered below is

$$\ddot{\mathbf{P}}/\omega_p^2 - \tau\dot{\mathbf{P}} + \mathbf{P} = \chi\mathbf{D}, \quad (1)$$

cf. Eq. (14) of the next section, where ω_p and τ are the resonance frequency and a damping time of the given resonance. This resonance may be overdamped ($\tau \gg 1/\omega_p$) or sharply resonating ($\tau \ll 1/\omega_p$), and the characteristic time τ_p , after which the polarization is no longer independent, is respectively τ and $2/(\omega_p^2\tau)$. The time scale for \mathbf{P} 's motion in the resonating case is $1/\omega_p$. (Note that going to a different resonance, τ_p will change, it therefore depends on the frequency of the external field.)

If the polarization \mathbf{P} is a specially slow variable, $\tau_p \gg \tau_{loc}$ (where τ_{loc} is around 10^{-10} s at usual temperatures and densities) we may increase the range of validity of the hydrodynamic electromagnetic theory, from $\omega \ll 1/\tau_p$ to $\omega \ll 1/\tau_{loc}$, by taking the energy as a local function also of \mathbf{P} , $\dot{\mathbf{P}}$, and derive the equation of motion for \mathbf{P} . There is quite a number of systems with a large τ_p : All electrorheological fluids have especially large τ_p , of the order of 10^{-4} s, but other complex fluids with large molecules and a permanent molecular dipole moment (such as nematic liquid crystals) also have a slow polarization. (Even the comparatively small water molecule with its permanent dipole moment has a τ_p around 10^{-9} s.) The hydrodynamic theory of dispersion presented below is unqualifiedly valid for these systems (though it should usually be enough to neglect $\ddot{\mathbf{P}}$ in the equation of motion, or equivalently, exclude $\dot{\mathbf{P}}$ as an additional variable.) We shall refer to this scenario as hydrodynamic dispersion.

Interestingly, essentially the same set of equations also accounts for a system in the ballistic regime, for quickly oscillating electric fields and polarizations, $1/\omega_p \ll \tau_{loc}$ —if we confine the theory to averaged quantities, such as what is the averaged force that a high frequency external field exerts on a volume element. (Note that this low resolution is quite sufficient for resolving the hydrodynamic responses to a high frequency field.) This scenario we shall refer to as ballistic dispersion. Clearly, we need to understand why the equations for hydrodynamic dispersion also work for ballistic disper-

sion, and what the restrictions are.

First, a coarse grained, hydrodynamic-type description is at all possible because the field variables \mathbf{P} , \mathbf{D} , and \mathbf{B} vary (due to the largeness of the light velocity) on macroscopic, hydrodynamic length scales. Second, most of the general principles used as input to consider hydrodynamic dispersion are also valid here. Especially, the total energy and momentum remain conserved. The one exception is local equilibrium, and equivalently, a defined value for the entropy density. The lack of these introduces some caveats with respect to temporal resolution and to dissipation. More specifically, they are the following:

(i) In the ballistic regime, the variables of the theory divide into two types, fast and slow. The field variables are fast, the rest are slow, including the densities of mass ρ , entropy s , total energy u , and total momentum \mathbf{g}^{tot} . The equations of motion of the field variables are highly accurate, resolving temporal increments much less than $1/\omega_p$; the actual hydrodynamic equations are coarse, with a resolution low compared to τ_{loc} . And because every differential equation, consistently, has a unique resolution, all field terms appearing in the slow, hydrodynamic equations need to be appropriately averaged.

(ii) The hydrodynamic theory of dispersion presented in the next section is clearly valid for ballistic dispersion in the transparent region. But because of the arguments listed below, it should remain valid even if field dissipation is strong.

Taking electromagnetic dissipation into account, the total, conserved energy divides into three parts,

$$U = U^{\text{mat}} + U^{\text{em}} + U^{\text{mic}}. \quad (2)$$

The first is the thermodynamic energy in the absence of an external field; the second is the additional energy in the presence of a field; and the third is the rest, the energy of all microscopic variables not given in the first two explicitly, $U^{\text{mic}}(x_1^2, x_2^2, \dots)$. The variables x_i are defined such that they vanish in local equilibrium, so they are irrelevant for the consideration of hydrodynamic dispersion. In the ballistic regime, U^{mic} is finite and serves as a transit hall: External energy is being fed continually into U^{em} , the electromagnetic dissipation excites some microscopic degrees of freedom x_i , and converts U^{em} into U^{mic} —which after the comparably long time of τ_{loc} becomes heat, $U^{\text{em}} \rightarrow U^{\text{mic}} \rightarrow \int T ds$. The rate at which U^{em} is lost is approximately $\dot{U}^{\text{em}} \approx -U^{\text{em}}/\tau_p$, the average time this energy stays in the transit hall is τ_{loc} , so $U^{\text{mic}} \approx (\tau_{loc}/\tau_p)U^{\text{em}}$. The right side translates into $(\tau_{loc}/\tau)U^{\text{em}}$ for the overdamped oscillation, and into $\frac{1}{2}(\omega_p^2 \tau_{loc} \tau)U^{\text{em}}$ for the resonating one. In the first case, we always have $U^{\text{mic}} \gg U^{\text{em}}$, and in the second we mostly do, rendering the transit hall usually large.

Including nonhydrodynamic variables such as \mathbf{P} leads to contributions $\sim \mathbf{P}$ and $\partial U/\partial \mathbf{P}$ in the energy and momentum flux, see the next section. If U^{mic} is nonzero, we would expect similar terms $\sim x_i$ and $\partial U/\partial x_i$. These we may neglect in the transparent region of vanishing dissipation, defined as the frequency regime where the stringent condition $\omega_p^2 \tau_{loc} \tau \ll 1$ holds, or equivalently $U^{\text{mic}} \ll U^{\text{em}}$, so U^{mic} and its contributions may be neglected. Outside these regimes, although circumstances are not as certain and more in need of a clari-

fying microscopic approach such as the Boltzmann theory, there are reasons why we may quite generally neglect U^{mic} .

While terms such as \mathbf{P}^2 and $\dot{\mathbf{P}}^2$, of macroscopic extent, coherently add up over many periods to yield slowly varying contributions, $\sim \langle \mathbf{P}^2 \rangle$ and $\langle \dot{\mathbf{P}}^2 \rangle$, in the momentum and energy flux, and thereby directly alter the slow, hydrodynamic variables, the quantities x_i are random and of microscopic spatial scales. So, instead of adding up, they further dissipate and degrade, to eventually turn into heat. On the coarse, hydrodynamic time scale, we may therefore lump $\langle U^{\text{mic}} \rangle$ into heat $\int T ds$, and $\langle (\partial/\partial t)U^{\text{mic}} \rangle$ into the heat production R . Then, clearly, U^{mic} may be neglected as an independent entity.

On a more operational level, the very criterion by which we have singled out \mathbf{P} and $\dot{\mathbf{P}}$ from the lot of microscopic degrees of freedom is their qualitatively different behavior. Given a certain energy content in the field \mathbf{D} and \mathbf{B} , there is a back and forth of energy flow between \mathbf{D} , \mathbf{B} , \mathbf{P} , and $\dot{\mathbf{P}}$ within each period; while the field energy that leaks into the other microscopic degrees of freedom is usually lost. In fact, for an overdamped resonance, it is (as mentioned) appropriate to exclude $\dot{\mathbf{P}}$ as an explicit variable, and consider it as one of the many ordinary microscopic degrees of freedom, as the energy leaked into $\dot{\mathbf{P}}$ is lost to the field. On the other hand, if a system involves more variables in the tidallike transfer of field energy, the present theory needs to be generalized to also include them—one example comes readily to mind: an independent magnetization.

Finally, a technical remark. Being a function also of the slow variables, U^{em} is, even without any dissipation (or electric charge), not conserved by itself, and the permeability ε will in general contain an imaginary part to account for this fact.

B. Equations of motion

The complete hydrodynamic theory of dispersion consists of a closed set of partial differential equations that governs the dynamics of the medium and the electromagnetic field. The structure of the equations is determined by general principles: the Maxwell equations, the Lorentz-Galilean transformation, the thermodynamic theory, and the relevant conservation laws.

Combining the two macroscopic energy densities,

$$U^{\text{Mac}} \equiv U^{\text{mat}} + U^{\text{em}} \quad (3)$$

we take it as a function of the entropy density s , mass density ρ , the electric and magnetic field \mathbf{D} and \mathbf{B} , the electric polarization \mathbf{P} , its canonical conjugate \mathbf{a} (that will turn out to be essentially $\sim \dot{\mathbf{P}}$), and the thermodynamic momentum density \mathbf{g} ,

$$dU^{\text{Mac}} = T ds + \mu d\rho + \mathbf{v} \cdot d\mathbf{g} + \mathbf{E} \cdot d\mathbf{D} + \mathbf{H} \cdot d\mathbf{B} + \mathbf{h} \cdot d\mathbf{P} + \mathbf{b} \cdot d\mathbf{a}, \quad (4)$$

where the thermodynamic momentum density \mathbf{g} is related to the total momentum density

$$\mathbf{g}^{\text{tot}} = \rho \mathbf{v} + (\mathbf{E} \times \mathbf{H})/c \quad (5)$$

through [7,10]

$$\mathbf{g} = \mathbf{g}^{\text{tot}} - \mathbf{D} \times \mathbf{B} / c. \quad (6)$$

As soon as the energy function U^{Mac} is known, the temperature T , chemical potential μ , velocity \mathbf{v} , field strengths \mathbf{E} and \mathbf{H} are also determined. (In accordance with [1], the polarization defined here is a rest frame quantity, $\mathbf{P} \equiv \mathbf{D}_0 - \mathbf{E}_0$.)

Isotropy of space results in the identity

$$\mathbf{E} \times \mathbf{D} + \mathbf{H} \times \mathbf{B} + \mathbf{h} \times \mathbf{P} + \mathbf{b} \times \mathbf{a} + \mathbf{v} \times \mathbf{g} = \mathbf{0}. \quad (7)$$

The Maxwell equations

$$\nabla \cdot \mathbf{B} = 0, \quad \dot{\mathbf{B}} = -c \nabla \times \mathbf{E}, \quad \nabla \cdot \mathbf{D} = \rho^e, \quad \dot{\mathbf{D}} = c \nabla \times \mathbf{H} - \rho^e \mathbf{v} \quad (8)$$

account for the motion of \mathbf{D} and \mathbf{B} . Here, the dot indicates partial temporal derivative $\partial/\partial t$ and ρ^e denotes the macroscopic charge density. The variables ρ , U , \mathbf{g}^{tot} are conserved, their equations of motion take the form

$$\dot{\rho} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (9)$$

$$\dot{U} + \nabla \cdot \mathbf{Q} = 0, \quad (10)$$

$$\dot{\mathbf{g}}_i^{\text{tot}} + \nabla_j (\Pi_{ij} - \Pi_{ij}^D) = 0, \quad (11)$$

where \mathbf{Q} is the total energy flux, and $(\Pi_{ij} - \Pi_{ij}^D)$ is the symmetric total momentum flux, or total stress tensor. The entropy is not conserved, and has a positive source R ,

$$\dot{s} + \nabla \cdot (s \mathbf{v} - \mathbf{f}^D) = R/T, \quad R \geq 0. \quad (12)$$

The dissipative part of entropy flux f^D describes especially thermoconduction, while Π_{ij}^D accounts primarily for viscosity-related phenomena.

The macroscopic variables \mathbf{P} and \mathbf{a} are governed by equations that are essentially of the Hamiltonian type,

$$\dot{\mathbf{P}} = \partial U / \partial \mathbf{a} = \mathbf{b}, \quad \dot{\mathbf{a}} = -\partial U / \partial \mathbf{P} = -\mathbf{h},$$

with some supplementary terms needed to ensure the proper transformation behavior, and to account for dissipation. First, the temporal derivative is replaced by the Galilean invariant operator that takes into account the effect of the local movement of the medium,

$$D_t = \partial_t + (\mathbf{v} \cdot \nabla) - \Omega \times, \quad (13)$$

where $\Omega \equiv \frac{1}{2} \nabla \times \mathbf{v}$. Second, a dissipative force \mathbf{h}^D is introduced in the equation for \mathbf{a} to account for electromagnetic dissipation that (in the linear case) is usually taken care of by an imaginary term in the electric permittivity ε . Third, the polarization is changed if the medium undergoes volume dilatation, as a term $-\mathbf{P}(\nabla \cdot \mathbf{v})$ appears in the equation of motion for \mathbf{P} ,

$$D_t \mathbf{P} = \mathbf{b} - \mathbf{P}(\nabla \cdot \mathbf{v}), \quad D_t \mathbf{a} = -\mathbf{h} - \mathbf{h}^D. \quad (14)$$

The dissipative terms are determined with the familiar method of irreversible thermodynamics: We first identify the entropy production as

$$R + \dot{U}^{\text{mic}} = \mathbf{f}^D \cdot (\nabla T) + \mathbf{h}^D \cdot \mathbf{b} + \Pi_{ij}^D v_{ij}, \quad (15)$$

where $v_{ij} = \frac{1}{2} (\nabla_i v_j + \nabla_j v_i)$. Then take the fluxes as proportional to the thermodynamic forces,

$$\begin{pmatrix} \Pi_{ik}^D \\ f_i^D \\ h_i^D \end{pmatrix} = \begin{pmatrix} \eta_{ikjl} & \alpha_{ikj} & \beta_{ikj} \\ \bar{\alpha}_{ijl} & \kappa_{ij} & \lambda_{ij} \\ \bar{\beta}_{ijl} & \bar{\lambda}_{ij} & \zeta_{ij} \end{pmatrix} \begin{pmatrix} v_{jl} \\ \nabla_j T \\ b_j \end{pmatrix} \quad (16)$$

(appropriate Onsager reciprocity relations are implied).

The energy flux is

$$\begin{aligned} Q_i &= (Ts + \mu \varrho + v_k g_k) v_i - T f_i^D - v_j \Pi_{ji}^D + c(\mathbf{E} \times \mathbf{H})_i + v_i(\mathbf{h} \cdot \mathbf{P}) \\ &\quad + \frac{1}{2} [\mathbf{v} \times (\mathbf{h} \times \mathbf{P} + \mathbf{b} \times \mathbf{a})]_i. \end{aligned} \quad (17)$$

(the last two terms were erroneously omitted from [1]). This expression may be rewritten as

$$Q_i = c(\mathbf{E}^0 \times \mathbf{H}^0)_i - f_i^D T + U^{\text{Mac}} v_i + (\Pi_{ij} - \Pi_{ij}^D) v_j - v_k g_k^{\text{tot}} v_i, \quad (18)$$

to see that the velocity-dependent terms do come from a Lorentz-Galilean transformation, discussed, e.g., in [8]. ($\mathbf{E}^0 \equiv \mathbf{E} + \mathbf{v} \times \mathbf{B} / c$ and $\mathbf{H}^0 \equiv \mathbf{H} - \mathbf{v} \times \mathbf{D} / c$ are the restframe fields.)

The stress tensor is symmetric and given as

$$\begin{aligned} \Pi_{ij} &= \frac{1}{2} [v_i g_j - E_i D_j - H_i B_j + (i \leftrightarrow j)] \\ &\quad + (Ts + \mu \rho + \mathbf{g} \cdot \mathbf{v} + \mathbf{H} \cdot \mathbf{B} + \mathbf{E} \cdot \mathbf{D} + \mathbf{h} \cdot \mathbf{P} - U^{\text{Mac}}) \delta_{ij}. \end{aligned} \quad (19)$$

Frequently, there are many different resonance frequencies of the polarization, not just the single one, given here as ω_p . This fact can be accounted for by introducing as many ‘‘subpolarizations,’’

$$\mathbf{D} - \mathbf{E} = \mathbf{P} = \sum \mathbf{P}_\alpha, \quad (20)$$

chosen such that the two squared order terms of the energy are diagonal,

$$U_0^{\text{em}} = \dots + \sum (\mathbf{P}_\alpha^2 / \chi_\alpha + \chi_\alpha \omega_\alpha^2 \mathbf{a}_\alpha^2) / 2 + \dots \quad (21)$$

Close to one resonance α , if it is well separated, as all the other subpolarizations are not excited, we may simply substitute \mathbf{P}_α for \mathbf{P} .

C. Some explicit expressions

Now, the above equations are rendered more explicit by an expansion of the energy function in the vector-variables \mathbf{D} , \mathbf{B} , \mathbf{P} , \mathbf{a} , and \mathbf{v} to third order, as this is sufficient for a comparison to the linear results by Brillouin and Pitaevskii's. For a magnetically inactive medium (i.e., taking the static magnetic permeability as 1), such an expansion yields

$$U^{\text{Mac}} = U^{\text{mat}} + \frac{1}{2} B^2 + \frac{1}{2} D^2 - \mathbf{D} \cdot \mathbf{P} + \frac{1}{2} P^2 / \chi + \frac{1}{2} \chi \omega_p^2 a^2 - \xi \mathbf{B} \cdot (\mathbf{P} \times \mathbf{a}) + \frac{1}{2} \rho v^2 + \mathcal{O}^4, \quad (22)$$

where \mathcal{O}^n denotes terms of n th or higher order in the vector variables $(\mathbf{D}, \mathbf{B}, \mathbf{P}, \mathbf{a}, \mathbf{v})$. The energy density in the absence of electromagnetic fields is $U^{\text{mat}}(s, \rho)$; the coefficient χ is related to the conventional static dielectric susceptibility $\chi' = P/E$ by $\chi'^{-1} = \chi^{-1} - 1$; ω_p^2 is the dielectric resonance frequency; ξ is connected to the magnetic cyclotron-frequency ω_B ; all these parameters are in principle functions of ρ and s .

Obtaining the differential form from Eqs. (22) and (6)

$$d(U^{\text{Mac}} - \mathbf{v} \cdot \mathbf{g}) = T ds + \mu d\rho + \mathbf{E} \cdot d\mathbf{D} + \mathbf{H} \cdot d\mathbf{B} + \mathbf{h} \cdot d\mathbf{P} + \mathbf{b} \cdot d\mathbf{a} - \mathbf{g} \cdot d\mathbf{v}, \quad (23)$$

we can derive the thermodynamically conjugate variables,

$$T = \frac{\partial U^{\text{mat}}}{\partial s} + \frac{P^2}{2} \frac{\partial \chi^{-1}}{\partial s} + \frac{a^2}{2} \frac{\partial \chi \omega_p^2}{\partial s} - \mathbf{B} \cdot (\mathbf{P} \times \mathbf{a}) \frac{\partial \xi}{\partial s} + \mathcal{O}^4, \quad (24)$$

$$\mu = \frac{\partial U^{\text{mat}}}{\partial \rho} + \frac{P^2}{2} \frac{\partial \chi^{-1}}{\partial \rho} + \frac{a^2}{2} \frac{\partial \chi \omega_p^2}{\partial \rho} - \mathbf{B} \cdot (\mathbf{P} \times \mathbf{a}) \frac{\partial \xi}{\partial \rho} - \frac{1}{2} v^2 + \mathcal{O}^4, \quad (25)$$

$$\mathbf{E} = \mathbf{D} - \mathbf{P} - \mathbf{v} \times \mathbf{M} + \mathcal{O}^3, \quad (26)$$

$$\mathbf{H} = \mathbf{B} - \xi (\mathbf{P} \times \mathbf{a}) + \frac{1}{c} \mathbf{v} \times \mathbf{P} + \mathcal{O}^3 \quad (27)$$

$$\mathbf{h} = \frac{1}{\chi} \mathbf{P} - \mathbf{D} - \xi \mathbf{a} \times \mathbf{B} + \frac{1}{c} \mathbf{B} \times \mathbf{v} + \mathcal{O}^3, \quad (28)$$

$$\mathbf{b} = \chi \omega_p^2 \mathbf{a} - \xi \mathbf{B} \times \mathbf{P} + \mathcal{O}^3. \quad (29)$$

Note that the magnetization $\mathbf{M} \equiv \mathbf{B} - \mathbf{H}$ as given in Eq. (27) is a term of order \mathcal{O}^2 . So the difference in the polarization $\mathbf{v} \times \mathbf{M}$, between the rest frame quantity $\mathbf{D}_0 - \mathbf{E}_0$ and the laboratory quantity $\mathbf{D} - \mathbf{E}$, is of order \mathcal{O}^3 . Within the accuracy of the above equations, it is therefore ignored.

Inserting Eq. (29) in Eq. (14), we obtain the expression

$$\mathbf{a} = \frac{1}{\chi \omega_p^2} [D_t \mathbf{P} + \mathbf{P}(\nabla \cdot \mathbf{v}) + \xi \mathbf{B} \times \mathbf{P}] + \mathcal{O}^3 \quad (30)$$

that may be used to eliminate \mathbf{a} in the above formulas, and write them instead with $\dot{\mathbf{P}}$. Especially, the magnetization in the rest frame and to lowest order is then $(\xi/\chi \omega_p^2) \mathbf{P} \times \dot{\mathbf{P}}$. The nonmagnetic magnetization is now seen to result from rotations of the polarization.

For a qualitative estimate of the coefficient ξ , envision electrons revolving around their ion centers [8]. Assuming the rotations to occur in phase, the magnetization associated with it is $(q_e n_e / 2m_e) \mathbf{L}$, where q_e and m_e are the charge and mass of the electrons, while n_e denotes their density. The angular moment of the electrons, $\mathbf{L} = m_e \mathbf{r}_e \times \dot{\mathbf{r}}_e$ (with \mathbf{r}_e the

radius of the circular motion) can also be written as $\mathbf{L} = (m_e / q_e^2 \omega_p^2) \mathbf{P} \times \dot{\mathbf{P}}$, because the polarization \mathbf{P} is $q_e n_e \mathbf{r}_e$. The attendant magnetization is $\mathbf{M} = (1/2q_e n_e) \mathbf{P} \times \dot{\mathbf{P}}$. So the coefficient ξ is

$$\xi = \chi \omega_p^2 / 2q_e n_e. \quad (31)$$

Particularly for an electron plasma, $\chi = 1$, and ω_p can be considered as the plasma frequency $(q_e^2 n_e / m_e)^{1/2}$. Equation (31) reduces to $\xi = q_e / 2m_e = -\omega_B / 2B$, with $\omega_B = -Bq_e / m_e$ the plasma cyclotron frequency.

III. MONOCHROMATIC APPROXIMATIONS

With the help of a closed dissipationless ac circuit, Pitae-vskiĭ obtained a number of important results on the effects of a high-frequency field in a medium [6]. Because of the special setup, the results are subject to certain restrictions. In order to compare our theory with his work, the same limits will be taken in our theory. Therefore, we shall consider a transparent medium exposed to a strictly monochromatic electric field:

$$\mathbf{E} = \frac{1}{2} \mathcal{E} e^{-i\omega t} + \text{c.c.}, \quad \dot{\mathcal{E}} = 0, \quad (32)$$

where \mathcal{E} is the constant amplitude and ω the frequency. From now on, we shall always assume that the medium is at rest, so any velocity-dependent terms will be discarded. Because only the electric properties are of interest, we also omit the quickly oscillating part of magnetic field in the medium, as in [6], though a strong, constant magnetic field is allowed to be present. If the material coefficients χ , ω_p^2 , ξ are constant with respect to time, the induction \mathbf{D} and polarization \mathbf{P} will also take the monochromatic form

$$\mathbf{D} = \frac{1}{2} \mathcal{D} e^{-i\omega t} + \text{c.c.}, \quad \mathbf{P} = \frac{1}{2} \mathcal{P} e^{-i\omega t} + \text{c.c.} \quad (33)$$

However, if χ , ω_p^2 , ξ , and \mathbf{B} are allowed to vary—slowly—via their dependence on the density or temperature, the fields \mathbf{D} and \mathbf{P} will become quasimonochromatic. In this case, we may still hold ω to be strictly constant, while allowing the amplitudes \mathcal{D} , \mathcal{P} to change slowly with time. The quasimonochromatic situation will be studied in Sec. III E.

In what follows, the dynamic equations given in the previous section will be investigated, under the preconditions mentioned above. We will show in detail the derivations of four formulas, all well known in the literature [4].

A. Permittivity

The frequency-dependent permittivity ε_{ij} is calculated from the equation of motion for \mathbf{P} . The expression is given by inserting Eqs. (30) and (28) into Eq. (14), taking the coefficients χ , ξ and the magnetic field \mathbf{B} as constants. Neglecting the velocity-dependent terms, we have

$$\mathbf{D} - \frac{\mathbf{P}}{\chi} - 2 \frac{\xi \mathbf{B} \times \dot{\mathbf{P}}}{\chi \omega_p^2} - \frac{\ddot{\mathbf{P}}}{\chi \omega_p^2} = \mathbf{0}. \quad (34)$$

If the fields \mathbf{E} , \mathbf{P} assume the monochromatic form of Eqs. (32) and (33), the above equation becomes

$$\left(1 - \frac{1}{\chi} + \frac{\omega^2}{\chi\omega_p^2}\right)\mathcal{P} + 2i\frac{\xi\omega}{\chi\omega_p^2}\mathbf{B}\times\mathcal{P} + \mathcal{E} = 0. \quad (35)$$

Solving it for \mathcal{P} , we obtain

$$\mathcal{P} = (\varepsilon_1 - 1)\mathcal{E} + \varepsilon_2(\mathbf{B}\cdot\mathcal{E})\mathbf{B} + i\varepsilon_3(\mathcal{E}\times\mathbf{B}) \quad (36)$$

with

$$\varepsilon_1 = 1 - \frac{\chi\omega_p^2(\omega^2 - \omega_p^2 + \chi\omega_p^2)}{(\omega^2 - \omega_p^2 + \chi\omega_p^2)^2 - \omega^2\omega_B^2}, \quad (37)$$

$$\varepsilon_2 = \frac{\chi\omega_p^2\omega^2\omega_B^2}{[(\omega^2 - \omega_p^2 + \chi\omega_p^2)^2 - \omega^2\omega_B^2](\omega^2 - \omega_p^2 + \chi\omega_p^2)B^2} \quad (38)$$

and

$$\varepsilon_3 = \frac{\chi\omega_p^2\omega\omega_B}{[(\omega^2 - \omega_p^2 + \chi\omega_p^2)^2 - \omega^2\omega_B^2]B}, \quad (39)$$

where

$$\omega_B = -2\xi B. \quad (40)$$

Using the fact that $\mathcal{P}_i = (\varepsilon_{ik} - \delta_{ik})\mathcal{E}_k$, we observe from Eq. (36) that the permittivity is

$$\varepsilon_{ik} = \varepsilon_1\delta_{ik} + \varepsilon_2B_iB_k + i\varepsilon_3\epsilon_{ikl}B_l, \quad (41)$$

where ϵ_{ikl} is the total antisymmetric tensor, $\epsilon_{123} = 1$. In the low frequency limit $\omega \rightarrow 0$, we have according to Eq. (37) $\varepsilon_1 = 1/(1 - \chi)$. Note that the imaginary term in (41) is not connected to dissipation. It is a purely reactive term. This can best be seen from its invariance under the time-reversal operation: $\omega \rightarrow -\omega$, $B_i \rightarrow -B_i$.

B. Energy density

Eliminating the quantity \mathbf{a} in the energy function (22) with the help of Eq. (30), we get, for a medium at rest and including terms of third order in the field

$$U = U^{\text{mat}} + \frac{B^2}{2} + \frac{D^2}{2} - \mathbf{D}\cdot\mathbf{P} + \frac{P^2}{2\chi} + \frac{\dot{P}^2}{2\chi\omega_p^2}. \quad (42)$$

Now consider the monochromatic case (32) and (33) and apply a time-averaged procedure denoted as $\langle \rangle$, the energy density is then given as

$$\langle U \rangle = U^{\text{mat}} + \frac{B^2}{2} + \frac{1}{4}\mathcal{E}_k\mathcal{E}_k^* + \frac{1}{4\chi}\left(1 + \frac{\omega^2}{\omega_p^2} - \chi\right)\mathcal{P}_k\mathcal{P}_k^*$$

or

$$\begin{aligned} \langle U \rangle &= U^{\text{mat}} + \frac{B^2}{2} + \frac{1}{4}|\mathcal{E}|^2 \\ &+ \frac{1}{4}\left(\frac{1}{\chi} + \frac{\omega^2}{\chi\omega_p^2} - 1\right)(\varepsilon_{km} - \delta_{km})(\varepsilon_{kn}^* - \delta_{kn})\mathcal{E}_m\mathcal{E}_n^*, \end{aligned} \quad (43)$$

where the asterisk denotes complex conjugation. Because in our work the permittivity is given by Eqs. (41) and (37)–(39), we can verify by direct computations that the equation

$$\frac{\partial\omega\varepsilon_{mn}}{\partial\omega} = \delta_{mn} + \frac{1}{\chi}\left(1 + \frac{\omega^2}{\omega_p^2} - \chi\right)(\varepsilon_{km} - \delta_{km})(\varepsilon_{kn}^* - \delta_{kn}) \quad (44)$$

holds for this form of permittivity. So the time-averaged energy density could be expressed as

$$\langle U \rangle = U^{\text{mat}} + \frac{B^2}{2} + \frac{1}{4}\frac{\partial\omega\varepsilon_{mn}}{\partial\omega}\mathcal{E}_m\mathcal{E}_n^*, \quad (45)$$

which is Brillouin's expression for the time-averaged energy density of the electric field [3].

C. Pitaevskii magnetization

Equation (27) shows that a magnetization could be induced dynamically in an electrically polarizable medium, although the static magnetic permeability is 1. Inserting Eq. (30) in Eq. (27) we obtain, for a medium at rest, the magnetization

$$\mathbf{M} = \frac{\xi}{\chi\omega_p^2}\mathbf{P}\times\dot{\mathbf{P}} + \mathcal{O}^3. \quad (46)$$

In the monochromatic approximation, the time-average of this magnetization is the same as that obtained by Pitaevskii [6]. Indeed, inserting Eqs. (32) and (33) in (46) we have

$$\begin{aligned} \langle \mathbf{M} \rangle_i &= \frac{i\omega\xi}{2\chi\omega_p^2}(\mathcal{P}\times\mathcal{P}^*)_i \\ &= \frac{i\omega\xi}{2\chi\omega_p^2}\epsilon_{ijk}(\varepsilon_{jm} - \delta_{jm})(\varepsilon_{kn}^* - \delta_{kn})\mathcal{E}_m\mathcal{E}_n^*. \end{aligned} \quad (47)$$

With the help of the expression of the permittivity (41), (37)–(39), one can show the validity of the equation

$$\frac{i\omega\xi}{\chi\omega_p^2}\epsilon_{ijk}(\varepsilon_{jm} - \delta_{jm})(\varepsilon_{kn}^* - \delta_{kn}) = \frac{1}{2}\frac{\partial\varepsilon_{nm}}{\partial B_i}. \quad (48)$$

So Eq. (47) possesses the form given by Pitaevskii

$$\langle \mathbf{M} \rangle = \frac{1}{4}\frac{\partial\varepsilon_{mn}}{\partial\mathbf{B}}\mathcal{E}_m^*\mathcal{E}_n. \quad (49)$$

D. Stress tensor

Inserting the expressions (22), (23), and (25) for U , T , μ into the stress (19) and eliminating the fields \mathbf{B} , \mathbf{a} , \mathbf{h} with the help of Eqs. (27), (28), and (30), the stress becomes

$$\begin{aligned} \Pi_{ij} = & \left[p_0 + \frac{H^2}{2} + \frac{E^2}{2} + \frac{1}{2} \left(\frac{1}{\chi} - 1 \right) P^2 - \frac{\dot{P}^2}{2\chi\omega_p^2} - \frac{\xi}{\chi\omega_p^2} \mathbf{H} \cdot (\mathbf{P} \times \dot{\mathbf{P}}) + \frac{P^2}{2} \left(\rho \frac{\partial}{\partial \rho} \frac{1}{\chi} \right) + \frac{1}{2} \frac{\dot{P}^2}{\chi^2 \omega_p^4} \rho \frac{\partial \chi \omega_p^2}{\partial \rho} - \mathbf{H} \cdot (\mathbf{P} \times \dot{\mathbf{P}}) \right. \\ & \left. \times \left(\rho \frac{\partial}{\partial \rho} \frac{\xi}{\chi \omega_p^2} \right) \right] \delta_{ij} - H_i H_j - \frac{1}{2} [E_i D_j + M_i H_j + (i \leftrightarrow j)] + \mathcal{O}^4, \end{aligned} \quad (50)$$

where p_0 is pressure of the medium in the case without electromagnetic fields,

$$p_0(\rho, s) = -U^{\text{mat}} + \rho \frac{\partial U^{\text{mat}}}{\partial \rho} + s \frac{\partial U^{\text{mat}}}{\partial s}. \quad (51)$$

Here, in order to avoid unnecessarily complicated formulas, we also neglect the entropy dependence of the parameters χ , ω_p^2 , ξ in (50). When the two electric fields take the monochromatic form (32) and (33), we obtain after the time-averaging procedure,

$$\begin{aligned} \langle \Pi_{ij} \rangle = & \left\{ p_0 + \frac{H^2}{2} + \frac{1}{4} \mathcal{E} \mathcal{E}^* - \frac{\omega^2 - \omega_p^2 + \chi \omega_p^2}{4\chi\omega_p^2} \mathcal{P} \mathcal{P}^* - \frac{i\omega\xi}{2\chi\omega_p^2} (\mathcal{P} \times \mathcal{P}^*) \mathbf{H} + \frac{1}{4\chi^2 \omega_p^2 H} [H(\omega^2 - \omega_p^2) \mathcal{P} \mathcal{P}^* - i\omega\omega_H (\mathcal{P} \times \mathcal{P}^*) \mathbf{H}] \rho \frac{\partial \chi}{\partial \rho} \right. \\ & \left. + \frac{1}{4\chi\omega_p^4 H} [H\omega^2 \mathcal{P} \mathcal{P}^* - i\omega\omega_H (\mathcal{P} \times \mathcal{P}^*) \mathbf{H}] \rho \frac{\partial \omega_p^2}{\partial \rho} + \frac{i\omega}{4\chi\omega_p^2 H} (\mathcal{P} \times \mathcal{P}^*) \mathbf{H} \rho \frac{\partial \omega_H}{\partial \rho} \right\} \delta_{ij} - H_i H_j - \frac{1}{8} (\mathcal{E}_i \mathcal{D}_j^* + \mathcal{E}_j \mathcal{D}_i^* + \text{c.c.}) \\ & - \frac{1}{2} (\langle M \rangle_i H_j + \langle M \rangle_j H_i). \end{aligned} \quad (52)$$

Because the difference between \mathbf{B} and \mathbf{H} [i.e., the Pitaevskii magnetization (49)] is of second order in \mathcal{E} , we can write the formula (36) for \mathcal{P} to the same accuracy as

$$\mathcal{P} = (\varepsilon_1 - 1) \mathcal{E} + \varepsilon_2 (\mathbf{H} \cdot \mathcal{E}) \mathbf{H} + i\varepsilon_3 (\mathcal{E} \times \mathbf{H}), \quad (53)$$

with

$$\varepsilon_1 = 1 - \frac{\chi \omega_p^2 (\omega^2 - \omega_p^2 + \chi \omega_p^2)}{(\omega^2 - \omega_p^2 + \chi \omega_p^2)^2 - \omega^2 \omega_H^2}, \quad (54)$$

$$\varepsilon_2 = \frac{\chi \omega_p^2 \omega^2 \omega_H^2}{[(\omega^2 - \omega_p^2 + \chi \omega_p^2)^2 - \omega^2 \omega_H^2] (\omega^2 - \omega_p^2 + \chi \omega_p^2) H^2} \quad (55)$$

and

$$\varepsilon_3 = \frac{\chi \omega_p^2 \omega \omega_H}{[(\omega^2 - \omega_p^2 + \chi \omega_p^2)^2 - \omega^2 \omega_H^2] H}, \quad (56)$$

where

$$\omega_H = -2\xi H. \quad (57)$$

Now inserting in the stress (52) the expression (53), it becomes

$$\begin{aligned} \langle \Pi_{ij} \rangle = & \left\{ p_0 + \frac{H^2}{2} - \frac{1}{4} \left(\rho \frac{\partial \varepsilon_{lm}}{\partial \rho} - \varepsilon_{lm} \right) \mathcal{E}_l^* \mathcal{E}_m \right\} \delta_{ij} - H_i H_j \\ & - \frac{1}{8} (\mathcal{E}_i \mathcal{D}_j^* + \mathcal{E}_j \mathcal{D}_i^* + \text{c.c.}) - \frac{1}{2} (\langle M \rangle_i H_j + \langle M \rangle_j H_i), \end{aligned} \quad (58)$$

where we have used the following equations valid for the ε_1 , ε_2 , ε_3 given by Eqs. (54)–(56),

$$\frac{\partial \varepsilon_1}{\partial \chi} = \frac{\omega_p^2 - \omega^2}{\chi^2 \omega_p^2} [(\varepsilon_1 - 1)^2 + \varepsilon_3^2 H^2] - \frac{2\omega\omega_H H}{\chi^2 \omega_p^2} \varepsilon_3 (\varepsilon_1 - 1), \quad (59)$$

$$\begin{aligned} \frac{\partial \varepsilon_2}{\partial \chi} = & \frac{\omega_p^2 - \omega^2}{\chi^2 \omega_p^2} [2\varepsilon_2 (\varepsilon_1 - 1) + \varepsilon_2^2 H^2 - \varepsilon_3^2] \\ & + \frac{2\omega\omega_H}{\chi^2 \omega_p^2 H} \varepsilon_3 (\varepsilon_1 - 1), \end{aligned} \quad (60)$$

$$\frac{\partial \varepsilon_3}{\partial \chi} = 2 \frac{\omega_p^2 - \omega^2}{\chi^2 \omega_p^2} \varepsilon_3 (\varepsilon_1 - 1) - \frac{\omega\omega_H}{\chi^2 \omega_p^2 H} [(\varepsilon_1 - 1)^2 + \varepsilon_3^2 H^2], \quad (61)$$

$$\frac{\partial \varepsilon_1}{\partial \omega_p^2} = -\frac{\omega^2}{\chi \omega_p^4} [(\varepsilon_1 - 1)^2 + \varepsilon_3^2 H^2] - \frac{2\omega \omega_H H}{\chi \omega_p^4} \varepsilon_3 (\varepsilon_1 - 1), \quad \frac{\partial \varepsilon_1}{\partial \omega_H} = \frac{2\omega H}{\chi \omega_p^2} \varepsilon_3 (\varepsilon_1 - 1), \quad (65)$$

$$\begin{aligned} \frac{\partial \varepsilon_2}{\partial \omega_p^2} &= -\frac{\omega^2}{\chi \omega_p^4} [2\varepsilon_2(\varepsilon_1 - 1) + \varepsilon_2^2 H^2 - \varepsilon_3^2] & \frac{\partial \varepsilon_2}{\partial \omega_H} &= -\frac{2\omega}{\chi \omega_p^2 H} \varepsilon_3 (\varepsilon_1 - 1), \\ &+ \frac{2\omega \omega_H}{\chi \omega_p^4 H} \varepsilon_3 (\varepsilon_1 - 1), & & \end{aligned} \quad (66)$$

$$\frac{\partial \varepsilon_3}{\partial \omega_p^2} = -\frac{2\omega^2}{\chi \omega_p^4} \varepsilon_3 (\varepsilon_1 - 1) - \frac{\omega \omega_H}{\chi \omega_p^4 H} [(\varepsilon_1 - 1)^2 + \varepsilon_3^2 H^2], \quad \frac{\partial \varepsilon_3}{\partial \omega_H} = \frac{\omega}{\chi \omega_p^2 H} [(\varepsilon_1 - 1)^2 + \varepsilon_3^2 H^2]. \quad (67)$$

Using Eq. (49) and the fact that $\mathcal{D}_i = \varepsilon_{im} \mathcal{E}_m$, we can also write the tensor (58) into the form

$$\begin{aligned} \langle \Pi_{ij} \rangle &= \left\{ p_0 + \frac{H^2}{2} - \frac{\mathcal{E}_i^* \mathcal{E}_m}{4} \left(\rho \frac{\partial \varepsilon_{lm}}{\partial \rho} - \varepsilon_{lm} \right) \right\} \delta_{ij} - H_i H_j - \frac{1}{4} \left\{ \varepsilon_1 \mathcal{E}_i \mathcal{E}_j^* + \varepsilon_2 (\mathbf{H} \mathcal{E}^*) \mathcal{E}_i H_j + \varepsilon_2 (\mathbf{H} \mathcal{E}) \mathcal{E}_i^* H_j \right. \\ &+ H_i H_j \left[\frac{\partial \varepsilon_1}{\partial H^2} \mathcal{E} \mathcal{E}^* + \frac{\partial \varepsilon_2}{\partial H^2} (\mathbf{H} \mathcal{E}) (\mathbf{H} \mathcal{E}^*) + i \frac{\partial \varepsilon_3}{\partial H^2} (\mathcal{E}^* \times \mathcal{E}) \mathbf{H} \right] + \frac{i \varepsilon_3}{2} [\mathcal{E}_i^* (\mathcal{E} \times \mathbf{H})_j + \mathcal{E}_i (\mathbf{H} \times \mathcal{E}^*)_j + H_i (\mathcal{E}^* \times \mathcal{E})_j] + (i \leftrightarrow j) \left. \right\}. \end{aligned} \quad (68)$$

If the following identity is noted

$$\mathcal{E}_i^* (\mathcal{E} \times \mathbf{H})_j + \mathcal{E}_i (\mathbf{H} \times \mathcal{E}^*)_j + H_i (\mathcal{E}^* \times \mathcal{E})_j = (\mathcal{E}^* \times \mathcal{E}) \cdot \mathbf{H} \delta_{ij}, \quad (69)$$

we finally obtain

$$\begin{aligned} \langle \Pi_{ij} \rangle &= \left\{ p_0 + \frac{H^2}{2} - \frac{\mathcal{E}_i^* \mathcal{E}_m}{4} \left(\rho \frac{\partial \varepsilon_{lm}}{\partial \rho} - \frac{\varepsilon_{lm} + \varepsilon_{lm}^*}{2} \right) \right\} \delta_{ij} - \frac{1}{4} [\varepsilon_1 \mathcal{E}_i \mathcal{E}_j^* + \varepsilon_2 (\mathbf{H} \mathcal{E}^*) (\mathcal{E}_i H_j + \mathcal{E}_j H_i) + \text{c.c.}] \\ &- H_i H_j - \frac{H_i H_j}{2} \left[\frac{\partial \varepsilon_1}{\partial H^2} \mathcal{E} \mathcal{E}^* + \frac{\partial \varepsilon_2}{\partial H^2} (\mathbf{H} \mathcal{E}) (\mathbf{H} \mathcal{E}^*) + i \frac{\partial \varepsilon_3}{\partial H^2} (\mathcal{E}^* \times \mathcal{E}) \mathbf{H} \right]. \end{aligned} \quad (70)$$

This agrees with Pitaevskii's stress tensor of a variable electric field in a liquid located in a strong magnetic field [6].

E. Time-dependent permittivity

In the previous subsections, we assumed that the parameters χ , ω_p^2 , ξ and the magnetic field \mathbf{B} are time independent. Consequently, the permittivity discussed in the Sec. III A is static. Now, we will abandon the restriction and allow χ , ω_p^2 , ξ , \mathbf{B} to change slowly with time. This case is naturally accounted for by Eqs. (14), (28), and (30), the equation of motion for \mathbf{P} ,

$$\frac{\partial}{\partial t} \left(\frac{\dot{\mathbf{P}}}{\chi \omega_p^2} + \frac{2\xi \mathbf{B}}{\chi \omega_p^2} \times \mathbf{P} \right) - \left(\frac{\partial}{\partial t} \frac{\xi \mathbf{B}}{\chi \omega_p^2} \right) \times \mathbf{P} + \left(\frac{1}{\chi} - 1 \right) \mathbf{P} - \mathbf{E} = \mathbf{0}. \quad (71)$$

Here, we again neglected the dissipation \mathbf{h}^D and considered a stationary medium: $\mathbf{v}(\mathbf{r}, t) \equiv \mathbf{0}$. Comparing the equation (71) with (34), we can see that the temporal variations of χ , ω_p^2 , ξ , \mathbf{B} give rise to additional terms, which result in a dynamic correction $\varepsilon_{ij}^{\text{dyn}}$ to the static dielectric permittivity obtained in Sec. III A. In other words, the relationship between the amplitude of polarization \mathcal{P} and that of electric field \mathcal{E} is no longer given by Eq. (36), but by

$$\mathcal{P}_i = (\varepsilon_{ij} - \delta_{ij}) \mathcal{E}_j + \varepsilon_{ij}^{\text{dyn}} \mathcal{E}_j, \quad (72)$$

where ε_{ij} is given by Eqs. (41) and (37)–(39). $\varepsilon_{ij}^{\text{dyn}}$ may be calculated by retaining a monochromatic electric field in Eq. (71): $\dot{\mathcal{E}} = 0$. Yet, because ε is now time-dependent, the amplitude of polarization \mathcal{P} will change with time, $\dot{\mathcal{P}}_i = \dot{\varepsilon}_{ij} \mathcal{E}_j$. And Eq. (71) becomes

$$\left(1 - \frac{1}{\chi} + \frac{\omega^2}{\chi\omega_p^2}\right)\mathcal{P} + 2i\frac{\xi\omega}{\chi\omega_p^2}\mathbf{B}\times\mathcal{P} + \mathcal{E} + \frac{\partial}{\partial t}\left(\frac{2i\omega}{\chi\omega_p^2}\hat{\boldsymbol{\varepsilon}}\mathcal{E} - \frac{2\xi\mathbf{B}}{\chi\omega_p^2}\times\hat{\boldsymbol{\varepsilon}}\mathcal{E}\right) - i\omega\left(\frac{\partial}{\partial t}\frac{1}{\chi\omega_p^2}\right)(\hat{\boldsymbol{\varepsilon}}+1)\mathcal{E} + \left(\frac{\partial}{\partial t}\frac{\xi\mathbf{B}}{\chi\omega_p^2}\right)\times(\hat{\boldsymbol{\varepsilon}}+1)\mathcal{E}=0, \quad (73)$$

where $\hat{\boldsymbol{\varepsilon}}$ is matrix notation of the permittivity (41). Solving for \mathcal{P} , we obtain

$$\boldsymbol{\varepsilon}_{ij}^{\text{dyn}} = (\varepsilon_{im} - \delta_{im}) \left[\frac{2i\omega}{\chi\omega_p^2} \varepsilon_{mj} + i\omega \left(\frac{\partial}{\partial t} \frac{1}{\chi\omega_p^2} \right) (\varepsilon_{mj} - \delta_{mj}) - \frac{2\xi}{\chi\omega_p^2} \epsilon_{mnl} B_n \dot{\varepsilon}_{lj} - \epsilon_{mnl} \left(\frac{\partial}{\partial t} \frac{\xi B_n}{\chi\omega_p^2} \right) (\varepsilon_{lj} - \delta_{lj}) \right]. \quad (74)$$

Together with the static permittivity derived before, this dynamic correction provides the expression for the full time-dependent permittivity.

To compare with [6], we decompose the dynamic correction (74) into a Hermitian and an anti-Hermitian part:

$$\boldsymbol{\varepsilon}_{ij}^{\text{dyn}} = {}'\boldsymbol{\varepsilon}_{ij}^{\text{dyn}} + i''\boldsymbol{\varepsilon}_{ij}^{\text{dyn}}. \quad (75)$$

Both matrices $'\boldsymbol{\varepsilon}^{\text{dyn}}$ and $''\boldsymbol{\varepsilon}^{\text{dyn}}$ may have complex elements, but must be Hermitian. In accordance with (75), we may also call them the real and imaginary parts of $\boldsymbol{\varepsilon}_{ij}^{\text{dyn}}$. Particularly, the imaginary part is

$$''\boldsymbol{\varepsilon}_{ij}^{\text{dyn}} = \frac{i}{2} (\boldsymbol{\varepsilon}_{ji}^{\text{dyn}*} - \boldsymbol{\varepsilon}_{ij}^{\text{dyn}}),$$

which can be also written as

$$\frac{\partial}{\partial t} \left[(\varepsilon_{im} - \delta_{im}) \left(\frac{\omega \delta_{mn} + i\xi \epsilon_{mkn} B_k}{\chi\omega_p^2} \right) (\varepsilon_{nj} - \delta_{nj}) \right], \quad (76)$$

here the fact $\varepsilon_{ij}^* = \varepsilon_{ji}$ is used. From the expression (41) and (37)–(39) for $\boldsymbol{\varepsilon}$, one can show that the equation

$$(\varepsilon_{im} - \delta_{im}) \left(\frac{\omega \delta_{mn} + i\xi \epsilon_{mkn} B_k}{\chi\omega_p^2} \right) (\varepsilon_{nj} - \delta_{nj}) = \frac{1}{2} \frac{\partial \varepsilon_{ij}}{\partial \omega} \quad (77)$$

is valid. So the imaginary part (76) of the dynamic contribution to the permittivity is

$$''\boldsymbol{\varepsilon}_{ij}^{\text{dyn}} = \frac{1}{2} \frac{\partial^2 \varepsilon_{ij}}{\partial \omega \partial t}. \quad (78)$$

This formula was first obtained in [6].

IV. DISCUSSIONS

Because both the dispersion and nonlinearity are accounted for, the hydrodynamic theory of dispersion sketched in Sec. II is a fairly complete theory for the dynamics of a fluid interacting with varying fields. The theory is derived by generalizing the hydrodynamic approach, but the result is consistent with the work of Pitaevskii, who starts from rather different physics.

Though not shown here, the present theory reduces to the hydrodynamic one [2], in the low frequency limit $\omega \rightarrow 0$. It is also in agreement with the Barash-Karpman's stress tensor derived for quasimonochromatic field (i.e., including the lowest-order effects of temporal variation of the field amplitude \mathcal{E}) [9]. All these features support the statement that the basic equations shown in Sec. II are correctly formulated, particularly the fundamental differential relation (4).

In our theory for dispersive media, Pitaevskii's magnetization appears as a consequence of circular motions of the polarization. In contrast to the conventional magnetization of atomic origin, Pitaevskii's magnetization is macroscopic. Because the circular motion of \mathbf{P} is usually accompanied by a rotating electric field, Pitaevskii's magnetization is less suitably generated by linearly polarized electromagnetic fields, as suggested in [6]. It is remarkable that the explanation to this phenomenon was given by the first nonlinear term in the expansion of the energy, Eq. (22). We expect it to be an important nonlinear effect, and to play a significant role in the nonlinear interaction between matter and intense laser lights.

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effects when leaving the equilibrium are linear in ω , they belong to $\boldsymbol{\varepsilon}''$, $\boldsymbol{\mu}''$ and are dissipative. Lack of dissipation therefore characterizes a frequency region disjunct from $\omega=0$.

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media. As a result, the ubiquitous factors of 4π , ϵ_0 , and μ_0 —the actual reason why we have to continually look up the formulas—simply vanish. To revert to MKSA ($\hat{E}, \hat{H} \dots$), employ

$$\begin{aligned} \hat{H} &= H/\sqrt{\mu_0}, & \hat{B} &= B\sqrt{\mu_0}, & \hat{E} &= E/\sqrt{\epsilon_0}, & \hat{D} &= D\sqrt{\epsilon_0}, \\ \hat{Q}_e &= Q_e\sqrt{\epsilon_0}, & \hat{j}_e &= j_e\sqrt{\epsilon_0}. \end{aligned}$$